Fredholm methods for billiard eigenfunctions in the coherent state representation

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We obtain a semiclassical expression for the projector onto eigenfunctions by means of the Fredholm theory. We express the projector in the coherent state basis, thus obtaining the semiclassical Husimi representation of the stadium eigenfunctions, which is written in terms of classical invariants: periodic points, their monodromy matrices, and Maslov indices.

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I. INTRODUCTION

The precise test of semiclassical approximations in the presence of chaos is of great interest to establish the limits of applicability of periodic orbit theory and its resummations. This test can be done in model systems both on approximations to the spectrum or to the stationary states. For the calculation of the spectrum the most efficient tool in this respect seems to be the spectral determinant [1,2] and several calculations [3] have demonstrated that, given enough periodic orbits, the spectrum can be accurately represented semiclassically. However, a more sensitive test—and still a great challenge—is the semiclassical representation of single eigenfunctions. This includes the study of the scar phenomena [4–10] and the eventual deviations from uniformity of eigenfunctions in accordance with the Berry-Voros hypothesis [11,12] and Schnirelman's theorem [13].

Just as for spectral problems, the use of Fredholm methods allows for the most efficient encoding of classical information in the calculation for single eigenfunctions [7,14]. In this paper we review these methods and apply them to the calculation of Husimi distributions of stadium eigenfunctions.

This paper is organized as follows. In Sec. II we review the Fredholm method for billiard eigenfunctions. Fredholm theory allows us to find the solution to certain type of integral or operator equations [15]. For billiards these methods can be applied to the boundary integral equation. In Sec. III we make the semiclassical approximation that is based on the approximation of the traces and powers of the propagator as sums over the periodic points of the underlying classical system. The propagator itself is taken as Bogomolny's $\mathbf{T}$ operator [1]. We choose the coherent state representation and obtain an expression for the semiclassical Husimi representation of the eigenfunctions in terms of classical invariants: periodic points, their monodromy matrices, and Maslov indices. In Sec. IV we apply this scheme for the stadium billiard. Our conclusions and perspectives are presented in Sec. V.

II. FREDHOLM FORMULAS FOR EIGENFUNCTIONS

Fredholm theory gives the solution to a certain class of integral equations, which can also be written as operator equations [15]. A Fredholm integral equation of the second type is

$$\chi(q)=\chi_0(q)+\lambda\int dq'\,\mathbf{T}(q',q)\chi(q'). \tag{1}$$

All the functions are defined in a finite domain. If the known functions $\chi_0(q)$ and $\mathbf{T}(q',q)$ are well behaved, the Fredholm alternative holds: there is a unique solution χ with the same analytic properties or the homogeneous equation ($\chi_0=0$) has a solution. There is a set of complex parameters λ_i for which the solution is not unique. In operator notation, the inverse of $(1-\lambda\mathbf{T})$ exists if $\lambda\neq\lambda_i$. In this case, this inverse can be written as

$$\frac{1}{1-\lambda\mathbf{T}}=\frac{\mathbf{M}(\lambda)}{D(\lambda)}, \tag{2}$$

where the operator $\mathbf{M}(\lambda)$ and the function $D(\lambda)$ are series in λ. If $\mathbf{T}$ is a compact operator, $D(\lambda)$ and $\mathbf{M}(\lambda)$ are entire in λ and, thus, absolutely convergent. The explicit form for the series expansion in terms of powers of $\mathbf{T}$ is given below. In what follows we apply this general theory assuming $\mathbf{T}$ to be unitary and of finite dimension N. Both assumptions are justified in the semiclassical limit for the quantization of billiards [1].

A. Secular equation

The k's eigenvalues are given by the secular equation $P(k)=\det[1-\mathbf{T}(k)]=0$. We can expand this determinant as

$$P(k)=\sum_{n=0}^{N}\beta_n(k), \tag{3}$$

where the coefficients $\beta_n(k)$ are related to the traces of $\mathbf{T}(k)$, $b_n(k)\equiv\operatorname{tr}\mathbf{T}^n(k)$, through

$$\beta_n(k)=-\frac{1}{n}\sum_{j=1}^{n}\beta_{n-j}(k)b_j(k). \tag{4}$$

Thus, knowledge of the traces up to a certain n_{max} implies the knowledge of the coefficients β_n up to the same n_{max}.

If $\mathbf{T}(k)$ is unitary, $P(k)$ is self-reversive, meaning that its coefficients satisfy

$$\beta_{N-j}(k)=(-1)^N\ \overline{\beta_j}(k)\det\mathbf{T}(k). \tag{5}$$

This condition alone forces the eigenvalues of $\mathbf{T}$ at fixed k to be symmetric with respect to the unit circle: if λ is an eigenvalue, then $1/\overline{\lambda}$ is an eigenvalue too. Of course, if $\mathbf{T}$ is unitary, then this condition is automatically satisfied but we can use it in our semiclassical approach to partially restore unitarity.

The contributions from coefficients β_i with $i>[(N+1)/2]$ can be expressed in terms of coefficients β_i with $i\leqslant[(N+1)/2]$. ($[x]$ is the integer part of x.) So, if N is even,

$$P(k)=\eta(k)+\det\mathbf{T}\,\overline{\eta}(k),$$

$$\eta(k)=\sum_{j=0}^{N/2-1}\beta_j(k)+\frac{1}{2}\beta_{N/2},\quad \det\mathbf{T}=\frac{\beta_{N/2}}{\overline{\beta}_{N/2}}. \tag{6}$$

If N is odd,

$$P(k)=\eta(k)-\det\mathbf{T}\,\overline{\eta}(k),$$

$$\eta(k)=\sum_{j=0}^{(N-1)/2}\beta_j(k),\quad \det\mathbf{T}=-\frac{\beta_{(N+1)/2}}{\overline{\beta}_{(N-1)/2}}. \tag{7}$$

As a consequence of the imposition of this symmetry on the operator $\mathbf{T}$ we obtain two advantages: only traces up to half the Heisenberg time $t_H=N$ are needed and the eigenvalues are constrained to lie on the unit circle or in symmetric pairs.

These formulas relate $P(k)$ with the traces of powers of $\mathbf{T}(k)$ which, in turn are related semiclassically to periodic orbits and to the smoothed density of states [16]. They have been tested extensively for the hyperbola billiard by Keating and Sieber [3].

B. Green function

To extend these methods to the calculation of eigenfunctions, we define a generalized Green function $\mathbf{G}(k)$:

$$\mathbf{G}(k)=\frac{\mathbf{T}(k)}{1-\mathbf{T}(k)}. \tag{8}$$

This operator has poles at the billiard eigenvalues $k=k_\nu$ and its residues are the projectors onto the corresponding eigenfunctions. It has a Fredholm expression as

$$\mathbf{G}(k)=\frac{\mathbf{T}(k)\mathbf{C}^t(1-\mathbf{T}(k))}{P(k)}, \tag{9}$$

where $\mathbf{C}^t(1-\mathbf{T}(k))$ is the transpose of the cofactor matrix of $1-\mathbf{T}(k)$ and, as in Eq. (2), has an expansion in powers of $\mathbf{T}(k)$.

It is then convenient to define a *normalized Green operator* as

$$\mathbf{g}(k)=\frac{\mathbf{G}(k)}{\mathrm{tr}[\mathbf{G}(k)]}, \tag{10}$$

where the singularities in the denominator have been eliminated. The normalized Green operator has the property $\mathbf{g}(k_\nu)=|\psi_\nu\rangle\langle\psi_\nu|$, where $|\psi_\nu\rangle$ is the eigenvector corresponding to eigenvalue k_ν. Then, we can write $\mathbf{g}(k)$ in the following way:

$$\mathbf{g}(k)=\frac{\mathbf{T}(k)C^t(1-\mathbf{T}(k))}{\mathrm{tr}[\mathbf{T}(k)C^t(1-\mathbf{T}(k))]}. \tag{11}$$

As the cofactor matrix can be expanded in powers of the propagator and as the propagator itself is unitary we write the normalized Green operator in terms of the powers of the propagator and their traces up to $N/2$ (if N is even):

$$\mathbf{g}(k)=\frac{\sum_{i=0}^{N/2-1}c_i(k)\mathbf{T}^{i+1}(k)-\det\mathbf{T}(k)\sum_{i=0}^{N/2-1}\overline{c_i}(k)\mathbf{T}^{\dagger i}(k)}{\sum_{i=0}^{N/2-1}c_i(k)\mathrm{tr}[\mathbf{T}^{i+1}(k)]-\det\mathbf{T}(k)\sum_{i=0}^{N/2-1}\overline{c_i}(k)\mathrm{tr}[\mathbf{T}^{\dagger i}(k)]}, \tag{12}$$

where the coefficients $c_i(k)$ are given by

$$c_i(k)=\sum_{n=i}^{N/2-1}\beta_{n-i}(k). \tag{13}$$

An analogous formula can be derived in case N is odd.

The coefficients $c_i(k)$ are dependent on the traces of $\mathbf{T}^n$ through Eqs. (4) and (13) and thus are independent of the chosen representation. On the other hand, the expression for the powers of the propagator will depend on the representation chosen for the calculation of the eigenfunctions. If the coordinate representation $|q\rangle$ is chosen, Eq. (12) relates the probability density $|\phi_\nu(q)|^2$ to the diagonal powers of the propagator $\langle q|\mathbf{T}^n(k)|q\rangle$. If the Weyl representation is chosen, then Eq. (12) gives the Wigner distribution of $|\phi_\nu\rangle$ in terms of the Weyl propagator [7,17]. Here we choose the coherent state representation to find the equivalent Husimi distribution.

We remark that Eq. (12) is a very compact and representation independent derivation of formulas that were previously very laboriously derived for the Wigner case. It prepares in an optimal way the grounds for the semiclassical approximation because its ingredients are all dependent on classical elements, namely periodic orbits, phase space volume, and generating function.

Using the fact that at $k=k_\nu$ the normalized Green function is the projector onto the corresponding eigenstate, we can obtain the Husimi distribution as

$$\mathcal{H}_{\psi_\nu}(z,\overline{z})=\frac{\langle z|\,\mathbf{g}(k_\nu)|z\rangle}{\langle z|z\rangle}=\frac{1}{\langle z|z\rangle}\;\frac{\sum_{i=0}^{N/2-1} c_i(k)\langle z|\mathbf{T}^{i+1}(k)|z\rangle-\det\mathbf{T}(k)\sum_{i=0}^{N/2-1}\overline{c_i}(k)\langle z|\mathbf{T}^{\dagger i}(k)|z\rangle}{\sum_{i=0}^{N/2-1} c_i(k)\mathrm{tr}[\mathbf{T}^{i+1}(k)]-\det\mathbf{T}(k)\sum_{i=0}^{N/2-1}\overline{c_i}(k)\mathrm{tr}[\mathbf{T}^{\dagger i}(k)]}. \tag{14}$$

This scheme was successfuly applied in simple quantum maps [18].

III. SEMICLASSICAL APPROXIMATION

Green's theorem allows us to reduce the Schrödinger equation for the billiard with Dirichlet boundary conditions to the following linear homogeneous equation for the normal derivative on the border $\phi(s)$,

$$\phi(s)=-2\oint ds'\,\phi(s')\mathbf{K}(s,s';k), \tag{15}$$

where the kernel is

$$\mathbf{K}(s,s';k)=\frac{-ik}{2}\cos\psi(s)\;\;H_1^{(1)}[k|\mathbf{r}(s)-\mathbf{r}'(s')|], \tag{16}$$

with k the wave number, $\psi(s)$ the angle between the normal at s and the line that connects $\mathbf{r}(s)$ with $\mathbf{r}'(s')$ (see Fig. 1), and $H_1^{(1)}$ the Hankel function of the first type and order one.

We introduce the wave function $\mu(s)$

$$\phi(p)=\frac{1}{ik}\sqrt{1-p^2}\,\mu(p), \tag{17}$$

with $\phi(p)$ and $\mu(p)$ the momentum representations of $\phi(s)$ and $\mu(s)$. This transformation makes the kernel symmetric and turns Eq. (15) to

$$\mu(s)=\oint\mathbf{T}(s',s;k)\mu(s')ds'. \tag{18}$$

The semiclassical theory of the kernel $\mathbf{T}(s',s;k)$ [1] is based on two fundamental properties: $\mathbf{T}$ is semiclassically unitary and has an effective dimension $N(k)=Lk/\pi$, where L is the length of the billiard. Moreover, the kernel is given by the generating function of the classical Birkhoff map [see Eq. (19)]. These properties have been extensively tested [19] and will be assumed in what follows.

Thus, we make the semiclassical approximation by taking $\mathbf{T}$ as Bogomolny's operator (19) and by evaluating all integrals by stationary phase approximation. The $\mathbf{T}$ operator for convex billiards in the plane, taking its border as the Poincaré section and using Birkhoff coordinates, is

$$\mathbf{T}(s',s;k)=\left(\frac{k}{2\pi i}\right)^{1/2}\left|\frac{\partial^2 l(s',s)}{\partial s\partial s'}\right|^{1/2}\exp\left(ikl(s',s)-i\frac{\pi}{2}\nu\right), \tag{19}$$

where the bounce map is generated by the arc length $l(s',s)$ between s and s'; ν is the Maslov index. The quantization condition is $\det[1-\mathbf{T}(k)]=0$.

In the semiclassical theory for the spectral determinant

$$P(k)=\det[1-\mathbf{T}(k)] \tag{20}$$

the approximate unitarity of $\mathbf{T}$ can be used efficiently to reduce the number of periodic orbits needed for the computation of the spectrum. Similar manipulations of the Fredholm formulas allow for the same reduction in the semiclassical calculation of single eigenfunctions of the billiard.

A. Semiclassical traces and determinant

First we need the traces. It is a well known fact that they adopt the following semiclassical expression [16]:

$$[b_n]_{scl}=\sum_{PO,n=n_pr}\frac{n_p}{|\det(I-M_p^r)|^{(1/2)}}\exp[ir(kl_p-\nu_p\pi/2)], \tag{21}$$

where the sum goes over all the primitive periodic orbits (PO's) of the billiard with period n_p, which must be a divisor of n, Maslov index ν_p, length l_p, and monodromy matrix M_p. The Maslov index can be interpreted geometrically: $\pi\nu_p$ is the angle swept by the unstable manifold of M_p along the PO.

The determinant of $\mathbf{T}$ can be obtained as [1]

$$[\det\mathbf{T}(k)]_{scl}=(-1)^N\exp[2\pi i\mathcal{N}(k)], \tag{22}$$

where $\mathcal{N}(k)$, the number of states between 0 and k, is

$$\mathcal{N}(k)=\frac{1}{4\pi}\mathcal{A}k^2-\frac{1}{4\pi}Lk, \tag{23}$$

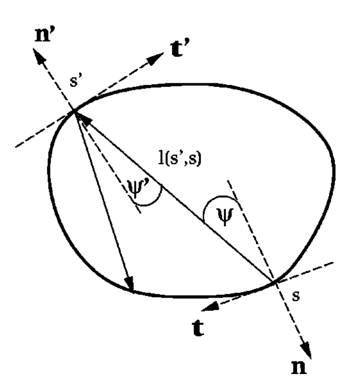

FIG. 1. Geometry of the billiard.

with $\mathcal{A}$ the area of billiard and L its length. These are the semiclassical ingredients needed for the calculation of the spectrum and of the coefficients $c_i(k)$.

B. Semiclassical propagator in coherent state representation

We first obtain the coherent state representation for one iteration of the bounce map,

$$\langle z'|\mathbf{T}|z\rangle = \int ds\, ds' \langle z'|s'\rangle\langle s'|\mathbf{T}|s\rangle\langle s|z\rangle. \tag{24}$$

That is to say,

$$\langle z'|\mathbf{T}|z\rangle = \left(\frac{k}{\pi\sigma^2}\right)^{1/2}\left(\frac{k}{2\pi i}\right)^{1/2} e^{(-i\pi\nu/2)} \times \int ds\, ds' \left|\frac{\partial^2 l}{\partial s \partial s'}\right|^{1/2} \exp[ik\Phi(s',s)], \tag{25}$$

where

$$\Phi(s',s) = \frac{i}{2}z'^2 + \frac{i}{2}z^2 \frac{i}{2\sigma^2}s'^2 - \frac{i\sqrt{2}}{\sigma}z's' + \frac{i}{2\sigma^2}s^2 - \frac{i\sqrt{2}}{\sigma}\bar{z}s + l(s',s), \tag{26}$$

and $z' = (q'/\sigma - i\sigma p')/\sqrt{2}$ and $z = (q/\sigma - i\sigma p)/\sqrt{2}$. The most important contributions come from those points s^* and s'^* that make stationary the phase Φ:

$$\frac{\partial\Phi}{\partial s}(s^*,s'^*) = \frac{i}{\sigma^2}s^* - \frac{i\sqrt{2}}{\sigma}\bar{z} + \frac{\partial l}{\partial s}(s^*,s'^*) = 0,$$

$$\frac{\partial\Phi}{\partial s'}(s^*,s'^*) = \frac{i}{\sigma^2}s'^* - \frac{i\sqrt{2}}{\sigma}z' + \frac{\partial l}{\partial s'}(s^*,s'^*) = 0. \tag{27}$$

The solution to Eq. (27) satisfying the reality conditions is

$$s^* = q, \quad \frac{\partial l}{\partial s}(s^*,s'^*) = -p$$

$$s'^* = q', \quad \frac{\partial l}{\partial s'}(s^*,s'^*) = p'. \tag{28}$$

This is a classical trajectory from (q,p) to (q',p'). Thus, the matrix element $\langle z'|\mathbf{T}|z\rangle$ will be nonzero only if z and z' are connected by the classical dynamics. Let us call these points z_c and z_c' and let us calculate the matrix element to next order in their neighborhoods, $\langle z_c' + \delta z'|\mathbf{T}|z_c + \delta z\rangle$. To this effect we expand $\Phi(s',s)$ in Eq. (25) to second order and, after some algebra,

$$\Phi(\delta s',\delta s) \approx l(q_c',q_c) + \left[-i\,\delta z'\bar{z}_c' - i\,\delta\bar{z}\bar{z}_c - \frac{i}{2}\bar{z}_c z_c - \frac{i}{2}\bar{z}_c' z_c'\right] + \left[\frac{i}{4}(\bar{z}_c'^2 - z_c'^2 - \bar{z}_c^2 + z_c^2)\right] + \left[\frac{i}{2}\delta z'^2 + \frac{i}{2\sigma^2}\delta s'^2 - \frac{i\sqrt{2}}{\sigma}\delta z'\delta s' + \frac{i}{2}\delta\bar{z}^2 + \frac{i}{2\sigma^2}\delta s^2 - \frac{i\sqrt{2}}{\sigma}\delta\bar{z}\delta s + \frac{1}{2}\left(\frac{\partial^2 l}{\partial s'^2}\delta s'^2 + 2\frac{\partial^2 l}{\partial s'\partial s}\delta s'\delta s + \frac{\partial^2 l}{\partial s^2}\delta s^2\right)\right], \tag{29}$$

where $\delta s = s - q_c$ and $\delta s' = s' - q_c'$. We change to new integration variables δs and $\delta s'$ and obtain

$$\langle z_c' + \delta z'|\mathbf{T}|z_c + \delta z\rangle \approx \left(\frac{k}{\pi\sigma^2}\right)^{1/2}\left(\frac{k}{2\pi i}\right)^{1/2}\exp(-i\pi\nu/2)\int d\,\delta s\, d\,\delta s' \left|\frac{\partial^2 l}{\partial s\partial s'}\right|^{1/2}\exp[ik\Phi(\delta s',\delta s)]. \tag{30}$$

We now insert Eq. (29) in Eq. (30). All terms are constant with respect to integration except the last one in square brackets. The resulting integral is the coherent state representation, with respect to $|\delta z\rangle$, of the linearized map, whose generating function is quadratic, which we have introduced in Eq. (A5):

$$\langle\delta z'|\mathbf{T}|\delta z\rangle = \frac{1}{\sqrt{s_c}}\exp\left[\frac{k}{2s_c}(-\bar{r}_c\delta z'^2 + 2\,\delta z'\,\delta\bar{z} + r_c\delta\bar{z}^2)\right], \tag{31}$$

where r_c y s_c are the matrix elements of the linearized map in complex coordinates. Finally we arrive at

$$\langle z_c' + \delta z'|\mathbf{T}|z_c + \delta z\rangle \approx \exp\left[\frac{-k}{4}(\bar{z}_c'^2 - z_c'^2 - \bar{z}_c^2 + z_c^2)\right]\exp\left[k\left(\delta z'\bar{z}_c' + \delta\bar{z}\bar{z}_c + \frac{1}{2}\bar{z}_c z_c + \frac{1}{2}\bar{z}_c' z_c'\right)\right]\exp\left(ikl - i\frac{\pi}{2}\nu\right)\langle\delta z'|\mathbf{T}|\delta z\rangle. \tag{32}$$

This result lets us evaluate the matrix element we were looking for, $\langle z|\mathbf{T}^n|z\rangle/\langle z|z\rangle$, which will be a sum of contributions of periodic points z_{pp} of period n in the semiclassical limit,

$$\frac{\langle z|\mathbf{T}^n|z\rangle}{\langle z|z\rangle} \approx \sum_{pp,n}\frac{\langle z_{pp} + \delta z|\mathbf{T}^n|z_{pp} + \delta z\rangle}{\langle z_{pp} + \delta z|z_{pp} + \delta z\rangle}. \tag{33}$$

To obtain the composition $\langle z_{pp} + \delta z|\mathbf{T}^n|z_{pp} + \delta z\rangle$ we use the expression (32) and the composition rule of Eq. (A6). Then

$$\frac{\langle z|\mathbf{T}^n|z\rangle}{\langle z|z\rangle}\approx\sum_{pp,n}\frac{1}{\sqrt{s_{pp}}}\lambda_{pp}\exp\left(ikl_{pp}-i\frac{\pi}{2}\nu_{pp}\right)\exp\left[\frac{k}{2s_{pp}}(-\overline{r_{pp}}\delta z^2+2\,\delta z\,\overline{\delta z}+r_{pp}\overline{\delta z^2})-k\,\delta z\,\overline{\delta z}\right], \tag{34}$$

where λ_{pp} can be calculated by Eq. (A7), $\nu_{pp}=n$ (because of the Dirichlet boundary conditions), and l_{pp} is the length of the PO starting from (q_{pp},p_{pp}). As we can see, the matrix element behaves as a Gaussian in the vicinities of the periodic point. This allows us to write the Husimi representation of the nth power of the propagator as a sum of contributions from periodic points of period n. Each term of the sum is a Gaussian packet in phase space whose parameters are related to the monodromy matrix in complex coordinates. A PO composed by n points will give *n different* contributions to this sum, due to the fact that the monodromy matrices at each point differ. However, the invariant properties of these matrices are the same and the usual Gutzwiller-Tabor trace formula [16] can be recovered by integration. Of course, a periodic point of period n will contribute also to the rn (r natural) powers of the propagator.

We should remark at this point that the different semiclassical representations of the propagator in terms of the corresponding generating function are only *semiclassically* equivalent and thus can give different results at finite N. This is not true for the calculation for the spectral determinant, whose semiclassical expression in terms of periodic orbits is the same in all representations. It is because of this that the different ways of computing eigenfunctions are not equivalent. For the calculation of $|\phi_\nu(s)|^2$ the closed (but not necesarily periodic) orbits are needed [1]. For the Wigner function calculation only periodic points are needed but each contribution is extended in phase space. In the present formalism we will obtain the Husimi distributions of eigenfunctions in terms of deformed localized Gaussians centered in the periodic points, constructed solely in terms of classical information.

C. Symmetries

Our system, the stadium billiard, has two discrete spatial symmetries: R_x and R_y, the two reflections with respect of the coordinate axes. These spatial symmetries in the domain reflect in the border and, thus, in the classical and quantum map on it. Their action on the Birkhoff coordinates of phase space (q,p) is

$$R_x(q,p)\to(L-q,-p),\quad R_y(q,p)\to\left(\frac{L}{2}-q,-p\right),$$

$$R_xR_y(q,p)\to\left(\frac{L}{2}+q,+p\right). \tag{35}$$

In order to have coherent states on the border with correct symmetries we need to project them using $\mathbf{R}_x$ and $\mathbf{R}_y$, the unitary representations of the symmetries $\mathbf{R}_x|x,y\rangle=|-x,y\rangle$ and $\mathbf{R}_y|x,y\rangle=|x,-y\rangle$. Then we define

$$|z_{\sigma_x\sigma_y}\rangle=\left(\frac{1+\sigma_x\mathbf{R}_x}{2}\right)\left(\frac{1+\sigma_y\mathbf{R}_y}{2}\right)\frac{|z\rangle}{\sqrt{\langle z|z\rangle}}, \tag{36}$$

where $\sigma_x,\sigma_y=\pm1$ and $\mathbf{R}_x$ and $\mathbf{R}_y$ move the center of the coherent state according to Eq. (35).

In this way, the diagonal matrix elements of the propagator in symmetrized coherent state representation are

$$\frac{1}{\langle z|z\rangle}\langle z|\mathbf{T}\left(\frac{1+\sigma_x\mathbf{R}_x}{2}\right)\left(\frac{1+\sigma_y\mathbf{R}_y}{2}\right)|z\rangle$$
$$=\frac{1}{4\langle z|z\rangle}(\langle z|\mathbf{T}|z\rangle+\sigma_x\langle z|\mathbf{TR}_x|z\rangle+\sigma_y\langle z|\mathbf{TR}_y|z\rangle$$
$$+\sigma_x\sigma_y\langle z|\mathbf{TR}_x\mathbf{R}_y|z\rangle). \tag{37}$$

We have already calculated $\langle z|\mathbf{T}^n|z\rangle$. We still have to calculate the other three contributions, $\langle z|\mathbf{T}^n\mathbf{R}_x|z\rangle$, $\langle z|\mathbf{T}^n\mathbf{R}_y|z\rangle$, and $\langle z|\mathbf{T}^n\mathbf{R}_x\mathbf{R}_y|z\rangle$. We can conclude using the results we have already obtained that each of them will be a sum of Gaussians centered in those points z that the dynamics connects with their symmetric partners, R_xz, R_yz, R_xR_yz, respectively. These points belong to PO's whose periods are $2n$, which are symmetric under the operations R_x, R_y, R_xR_y, respectively. The increment $R\,\delta z$ with respect to Rz ($R\equiv R_x,R_y,R_xR_y$) is related to the increment δz with respect to z through

$$R\,\delta z=t_R\delta z,\quad t_R=\begin{cases}-1 & \text{if } R=R_x\\ -1 & \text{if } R=R_y\\ 1 & \text{if } R=R_xR_y.\end{cases} \tag{38}$$

Thus we arrive at

$$\frac{\langle z|\mathbf{T}^n\mathbf{R}|z\rangle}{\langle z|z\rangle}\approx\sum_{pp,2n}\frac{1}{\sqrt{s_{pp}}}\lambda_{pp}\exp\left(ikl_{pp}-i\frac{\pi}{2}\nu_{pp}\right)$$
$$\times\exp\left[\frac{k}{2s_{pp}}(-\overline{r_{pp}}\delta z^2+2t_R\delta z\,\overline{\delta z}\right.$$
$$\left.+r_{pp}\overline{\delta z^2})-k\,\delta z\,\overline{\delta z}\right], \tag{39}$$

where the sum goes over the periodic points of period $2n$ that belong to PO's symmetric under R. The quantities s_{pp}, r_{pp}, l_{pp}, ν_{pp}, and λ_{pp} are calculated along the trajectory that connect z to Rz, i.e., half PO.

IV. SEMICLASSICAL EIGENFUNCTIONS FOR THE STADIUM

We use the semiclassical approach we introduced above for the stadium billiard. We choose odd-odd symmetries ($\sigma_x=\sigma_y=-1$) and $\sigma=2$. We have periodic points with desymmetrized period up to 8 (around 800). We have used the symbolic dynamics developed by Biham and Kvale [20] to obtain them. The wave number k is related to the maximum period used in the expansion (14) by $P(k)=(L/2\pi)k$

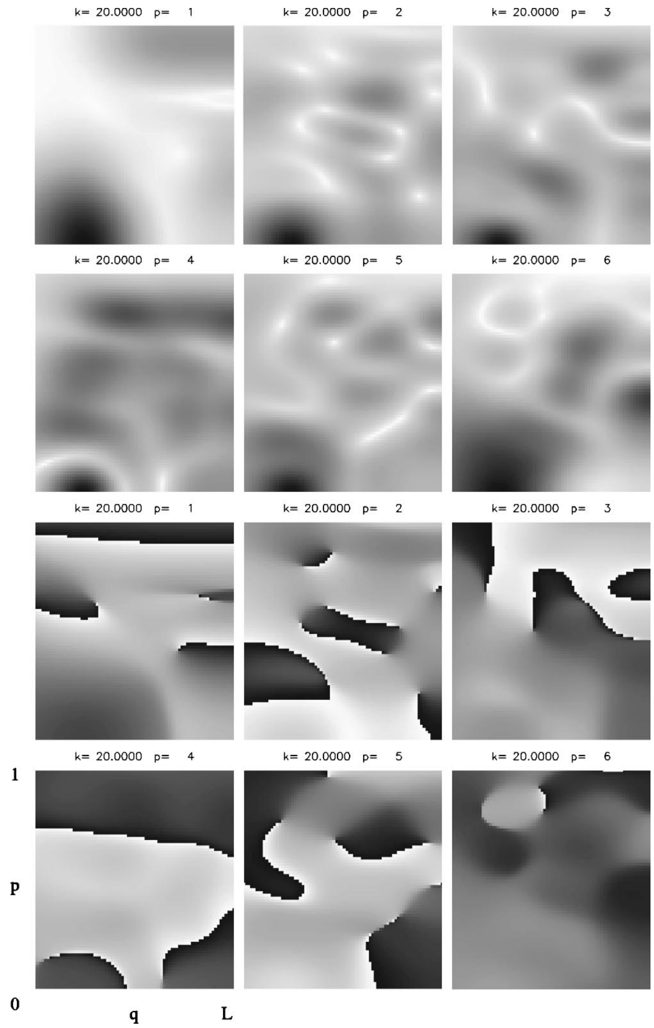

FIG. 2. Phase space representations for the first six powers of Bogomolny's operator. For each power we show modulus (rows 1 and 2) and phase (rows 3 and 4).

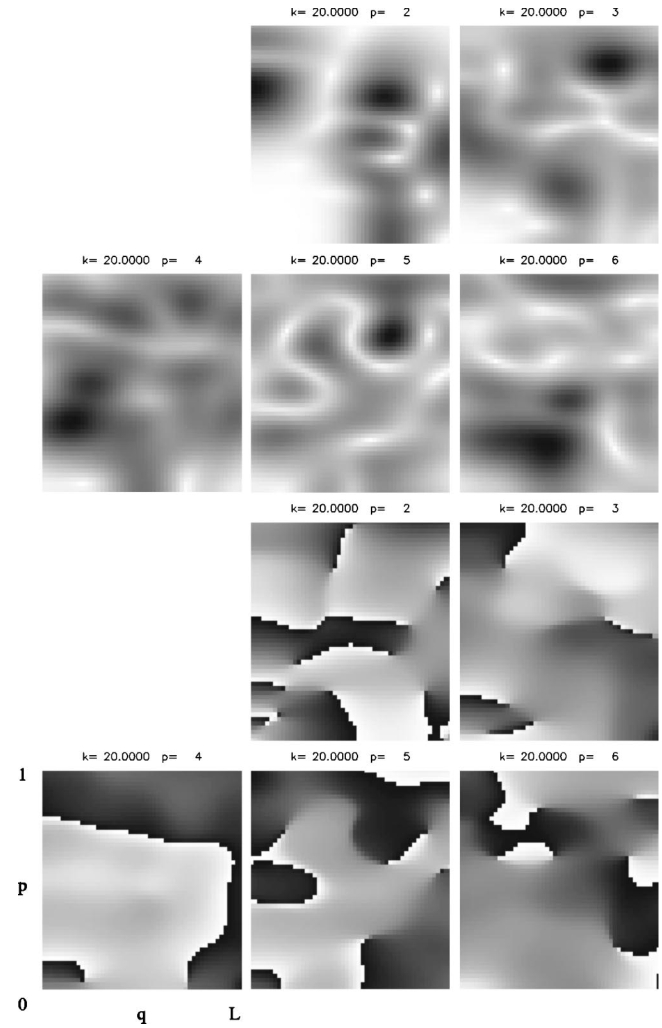

FIG. 3. Semiclassical phase space representations for the first six powers of Bogomolny's operator. For each power we show modulus (rows 1 and 2) and phase (rows 3 and 4).

$\approx 0.4k$. In this way we can obtain semiclassical approximations of eigenfunctions of wave number $k \leqslant 20$.

In Fig. 2 we show the phase space representations of the first six powers of Bogomolny's **T** operator for $k=20$; in Fig. 3 we show the semiclassical approximations. We see that the exact representations show global maxima in the bouncing ball region that cannot be reproduced semiclassically for the lowest powers. However, the overall semiclassical behavior is very close to that of the exact representations. (Because of the symmetries we chose, we have no semiclassical approximation to the first power of the operator because the contribution of the only periodic point of period 1 is zero.)

We select two energy ranges: $k \in [19.1,20.0]$ and $k \in [20.5,21.3]$. There are four eigenenergies in each of these ranges. We show the absolute value of the secular determinant, $|P(k)|$, for each of them in Figs. 4 and 5. The full line is the semiclassical approximation, the dashed line is the secular determinant for Bogomolny's operator. The vertical lines are the exact quantum k eigenvalues calculated by the scaling method [21]. We see a good approximation when we use the periodic point expansion. The agreement shows that in this region the spectrum is well represented semiclassically. (The discontinuities come from the change in dimension of the operator.)

To keep the method consistent we evaluated the semiclassical Husimi expansion in those values of k that minimize the semiclassical secular determinant. We see in Figs. 6 and 7 the exact eigenfunctions (first column) and their corresponding semiclassical Husimi representations (second column) obtained as the real part of Eq. (14). The global behavior is well reproduced; however, the finer details are hard to mimic. The bouncing ball region is problematic: in some functions (e.g., $k=21.16$) some probability leaks to this region. Probably PO's with longer periods that approximate the bouncing ball orbits could make a better picture for this region.

One of the advantages of formula (12) for the projector is that it has no singularities between eigenvalues. It is possible to study continually its behavior as a function of k in order to see its sensitivity to changes in k. Some properties of the exact distribution as a function of k are as follows [22]:

(i) The distribution is positive at the eigenvalues k_n.

(ii) The distribution has $N(k)$ zeros at eigenvalues k_n.

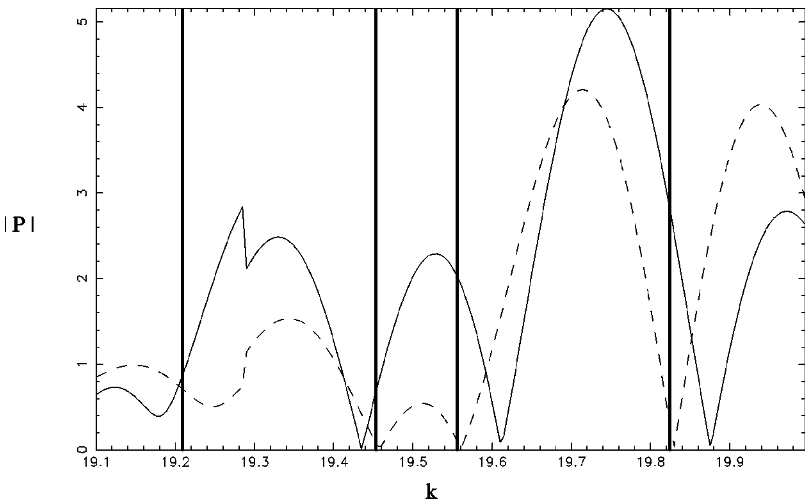

FIG. 4. Secular determinant. In full line we show the PO's approximation; in dashed line, the exact by using Bogomolny's operator. For the semiclassical approximation we summed up to period 8.

These two properties follow from the fact that $\mathcal{H}_{\psi_\nu}(z,\overline{z})$ is the modulus of an analytic function. The distributions between eigenvalues can become negative. In particular, it can be shown that at the value of k that maximizes $P(k)$, the distribution is constant. This property can be used to control the semiclassical approximations.

In Figs. 8 and 9 we show the behavior of the distribution $\langle z|\mathbf{g}(k)|z\rangle/\langle z|z\rangle$ between the semiclassical eigenvalues $k=19.18$ and $k=19.38$. When $k=19.18$ the distribution is positive and has well defined minima that approach zero. As we move away from the eigenvalue, the distribution changes smoothly. Initially it moves away from the plane $g=0$ in the positive direction, then it comes back and turns negative. During this ''evolution'' it flattens visibly and we cannotdiscern its features. At $k=19.32$, aproximately the maximum of the secular determinant, see Fig. 4, the distribution is constant. At the semiclassical eigenvalue $k=19.38$ the distribution is positive again with well defined minima.

We can see from Figs. 6 and 7 that the semiclassical approximation is relatively good. It is not trivial to obtain a positive defined distribution with N zeroes adding several hundreds of Gaussians, each with its phase and deformation.

V. CONCLUSIONS

Using Fredholm theory we have given a very compact and representation independent derivation of the projector on a single eigenfunction for unitary quantum maps. Expressing the projector in the coherent state basis we wrote a semiclassical expression for the Husimi distributions of the billiard's eigenfunctions. Each periodic point contributes with a Gaussian centered in it whose parameters are calculated only with classical information. We should not underestimate the difficulties and complexities inherent to this method. Hundreds of Gaussian contributions have to conspire to make a positive definite distribution with N zeros that approximate the quantum Husimi distributions.

The projector (12) can be represented in coordinate space. We obtain $\langle q|\psi\rangle\langle\psi|q\rangle$, whose semiclassical approximation can be directly compared to the probability density in the section. This representation has an additional difficulty, since the semiclassical approximation is written as a sum over closed trajectories, periodic or not, in configuration space. Those that are not periodic are more in number and more difficult to find. Anyway, we can apply our scheme for Bogomolny's $\mathbf{T}(q',q)$ operator and compare the results with the exact quantum calculation. In Fig. 10 we see that the approximation is excellent at this level.

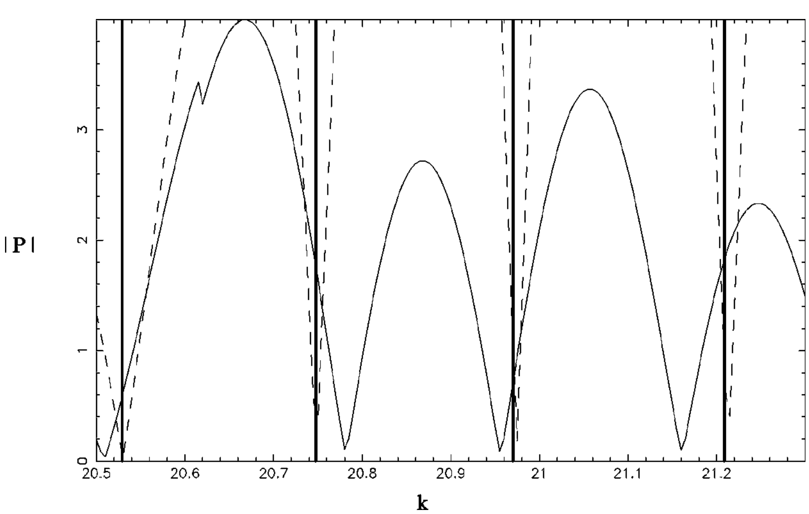

FIG. 5. Secular determinant. In full line we show the PO's approximation; in dashed line, the exact by using Bogomolny's operator. For the semiclassical approximation we summed up to period 8.

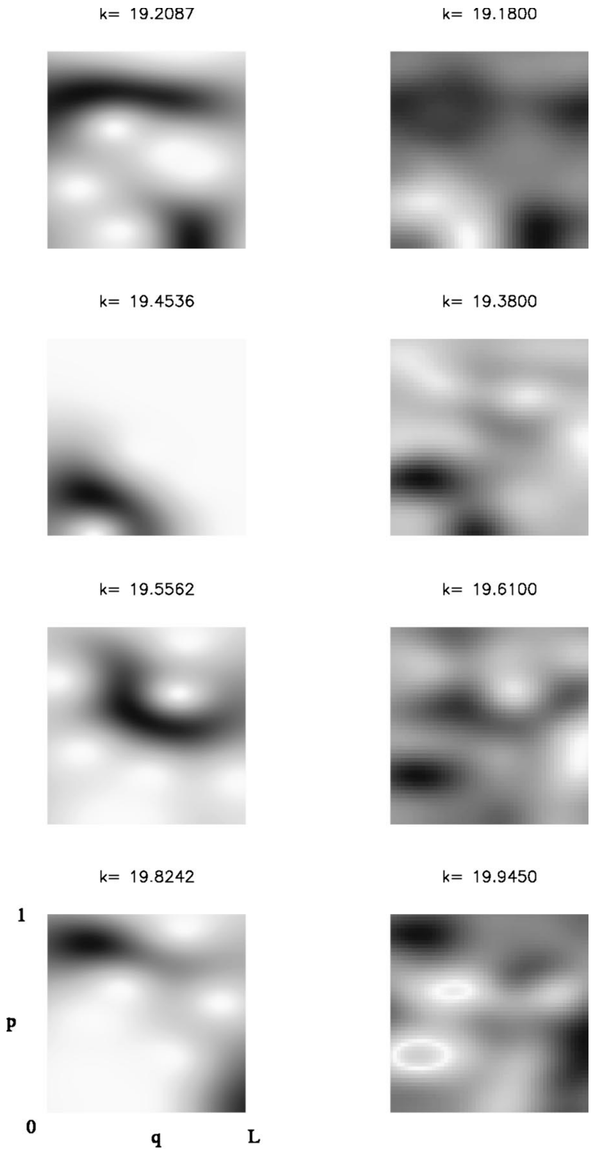

FIG. 6. Husimi representations of stadium eigenfunctions (left panels) and their semiclassical approximations (right panels) for the energy range of Fig. 4. For the semiclassical approximation we summed up to period 8.

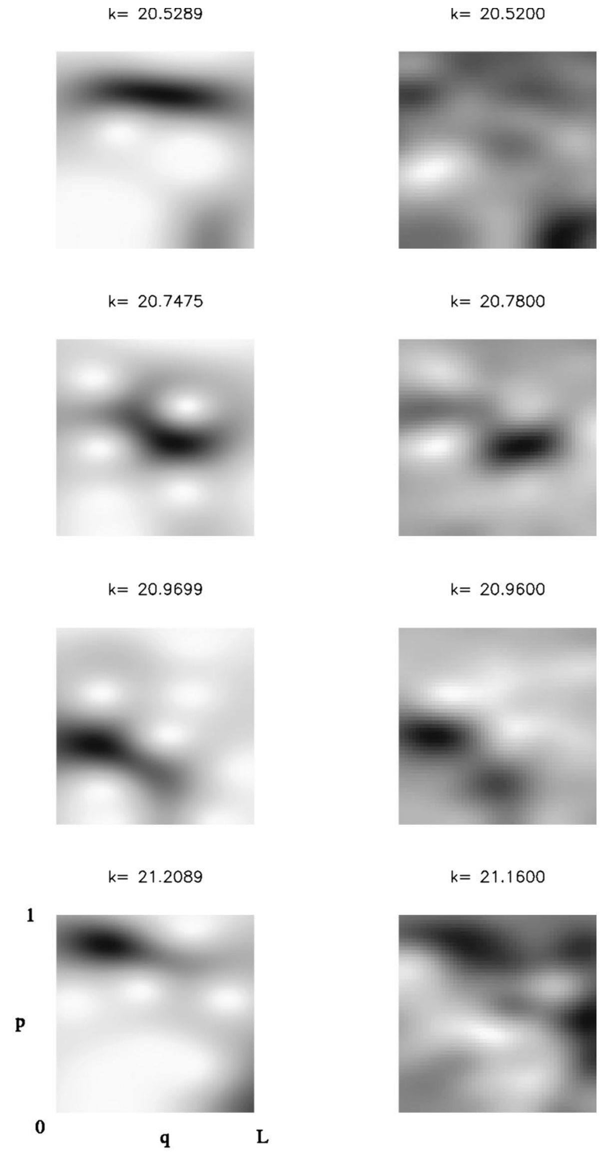

FIG. 7. Husimi representations of stadium eigenfunctions (left panels) and their semiclassical approximations (right panels) for the energy range of Fig. 5. For the semiclassical approximation we summed up to period 8.

The maximum period P in the expansions is related to the energy in the way $P \approx 0.4k$. Due to the exponential proliferation of orbits in chaotic systems, the method cannot be applied for arbitrarily high energies. The measure of this proliferation is the topological entropy W which relates the number N_P of PO's of a given period P with the period itself, $N_P = \exp(WP)$ [23]. For the stadium, $W \approx 0.94$. Then, for $k \approx 100$ we need PO's of periods up to $P = 40$, whose number is $N_P \approx \exp(0.94 \times 40) \approx 10^{17}$. The Fredholm method we developed is a first step and shows that the eigenfunctions can be described as expansions in terms of the periodic points of the underlying classical system. It eliminates the divergencies associated that the schemes based on smoothings in energy have. However, the exponential divergence of periodic orbits poses a serious practical problem, as discussed in the previous paragraph. This method can only become practical for large k_ν if some way of selecting a few ''important'' orbits at each value of k can be developed. Some results in this direction have been obtained by Vergini and Carlo [21,24,25].

APPENDIX A: COMPLEX PHASE SPACE

We introduce the following symplectic transformation Z, depending upon parameter σ, acting on a point of classical phase space (q,p) [26]:

$$\begin{pmatrix} z \\ p_z \end{pmatrix} = Z \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2}\sigma & -i\sigma/\sqrt{2} \\ -i/\sqrt{2}\sigma & \sigma/\sqrt{2} \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}. \qquad \text{(A1)}$$

Imposing reality conditions on the inverse transformation we see that $\overline{z} = ip_z$. A linear transformation $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in (q,p) phase space has a representation M_z in (z,p_z) phase space by conjugation with Z,

$$M_z = ZMZ^{-1} = \begin{pmatrix} \overline{s} & -ir \\ i\overline{r} & s \end{pmatrix} \quad \text{with} \quad \begin{cases} s = \dfrac{1}{2}\left[(a+d) - i\left(\dfrac{b}{\sigma^2} - \sigma^2 c\right)\right] \\ r = \dfrac{1}{2}\left[(d-a) + i\left(\dfrac{b}{\sigma^2} + \sigma^2 c\right)\right]. \end{cases} \tag{A2}$$

This (z,p_z) phase space allows a passage to quantum mechanics. This is done in a Hilbert Bargmann space by introducing operators $\mathbf{z}$ and $\mathbf{p_z}$ that satisfy the conmutator relations

$$[\mathbf{z},\mathbf{p_z}] = i\hbar, \quad [\mathbf{z},\mathbf{z}] = [\mathbf{p_z},\mathbf{p_z}] = 0. \tag{A3}$$

Any vector $|\psi\rangle$ in Hilbert space can be represented in this new space as $\langle z|\psi\rangle = \int dq \langle z|q\rangle\langle q|\psi\rangle$, where the coherent states are $\langle z|q\rangle = [1/(\pi\hbar\sigma^2)]^{1/4}\exp[-(1/\hbar)(z^2/2 + q^2/(2\sigma^2) - \sqrt{2}zq/\sigma)]$. The scalar product is $\langle\psi_1|\psi_2\rangle = \int \overline{\psi_1}(z)\psi_2(z)d\mu(z)$ with norm $d\mu(z) = (1/\pi)\exp(-z\overline{z}/\hbar)d\,\mathrm{Re}(z)d\,\mathrm{Im}(z)$.

We now define the Husimi representation of a vector ψ as

$$\mathcal{H}_\psi(z) \equiv \frac{|\langle z|\psi\rangle|^2}{\langle z|z\rangle}. \tag{A4}$$

It is a real positive function for every z in the complex plane.

The representation of a linear symplectic transformation M in phase space in terms of a unitary operator of Hilbert Bargmann space is [27]

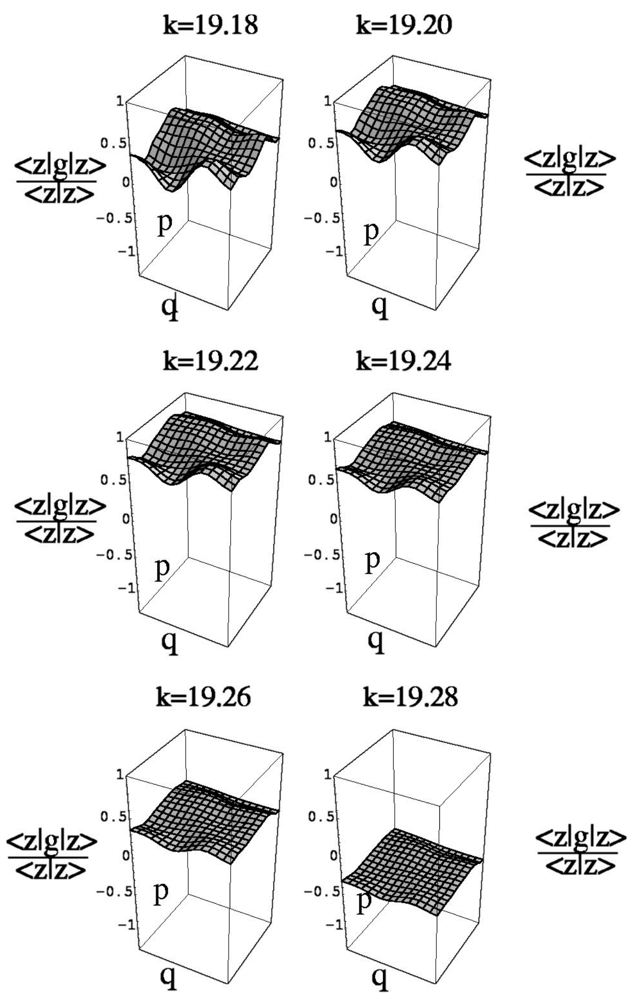

FIG. 8. Variation of the normalized Green function in coherent state representation between $k=19.18$ and $k=19.28$. $k=19.18$ is a semiclassical eigenvalue.

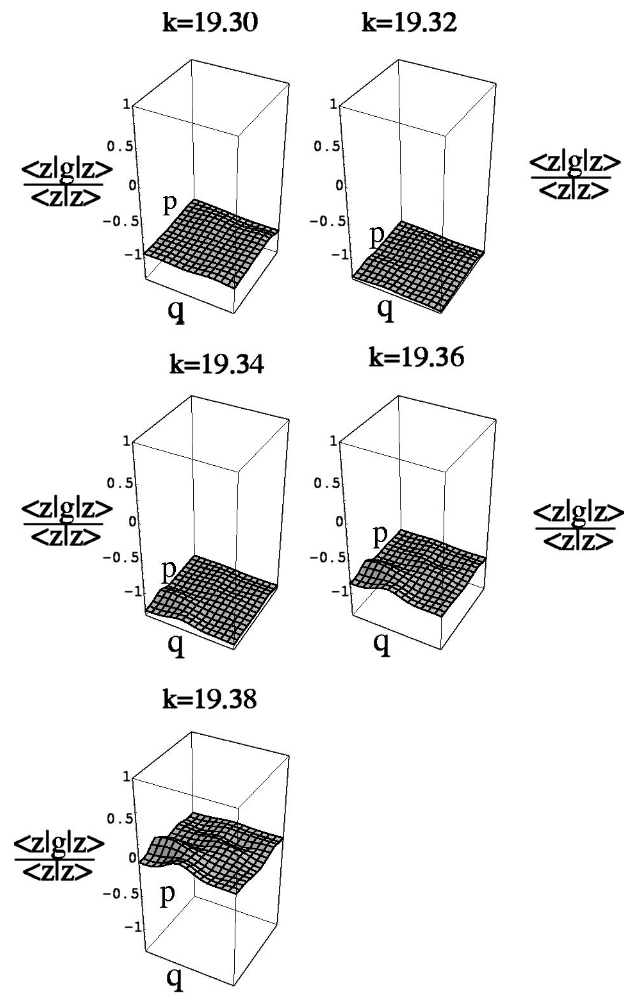

FIG. 9. Continuation of Fig. 8. Variation of the normalized Green function in coherent state representation between $k=19.28$ and $k=19.38$. $k=19.38$ is a semiclassical eigenvalue.

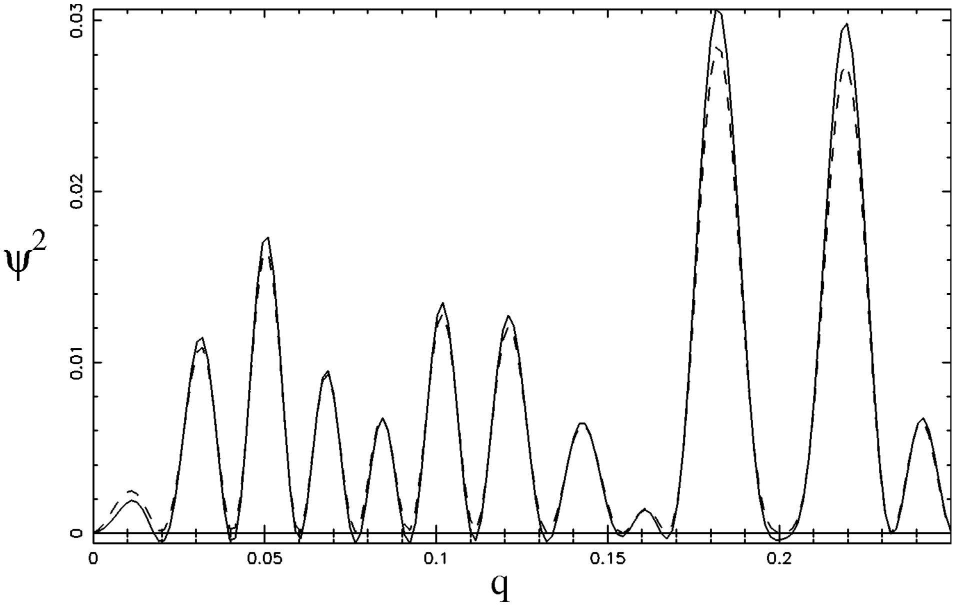

FIG. 10. Exact eigenfunction (dashed line) for $k=20.5289$ and its approximation by using Bogomolny's operator in the normalized Green function (full line).

$$M=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \langle z' | \mathbf{U}(M) | z \rangle$$

$$= \frac{1}{\sqrt{|\bar{s}|}} \exp\left(\frac{-i}{2}\arg(\bar{s})\right) \exp\left[\frac{k}{2\bar{s}}(-\bar{r}z'^2+2z'\bar{z}+r\bar{z}^2)\right]. \tag{A5}$$

This representation is up to a phase [27,28] and its composition law is

$$\int \langle z' | \mathbf{U}(M_1) | z \rangle \langle z | \mathbf{U}(M_2) | z'' \rangle d\mu(z)$$

$$= \lambda(M_1,M_2,M_1M_2) \langle z' | \mathbf{U}(M_1M_2) | z'' \rangle, \tag{A6}$$

with

$$\lambda(M_1,M_2,M_1M_2) = \exp\left\{ \frac{i}{2} \left[\arg(\bar{s}) - \arg(\bar{s}_1) - \arg(\bar{s}_2) - \arg\left(\frac{\bar{s}}{\bar{s}_1 \bar{s}_2}\right) \right] \right\} = \pm 1. \tag{A7}$$

The accumulated phase due to succesive transformations leads to the Maslov index of the trajectory.

In case the phase space shows periodicity in coordinate or momentum, we have to periodize the coherent states as in [29].

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